Let us examine the point of intersection of Π_σ and σ_H $(v,\ v_0)$. Corresponding to the possible shock crossings will be those points M_1 of the intersection of Π_σ and $\sigma_H(v, v_0)$ for motion to which along the ray from M_0 there are no points of σ_H to the left of the ray. Furthermore, if M lies in the shear plasticity domain with compression, then the condition $Y\ (M) < 0$ should be satisfied, where the calculations should be carried out for shear elasticity. If this condition is satisfied, then the shock crossing is unique. In particular, it is satisfied if (6) is satisfied. If $Y\ (M) \geqslant 0$ for shear elasticity, then a shock crossing of the same intensity is possible with the change in the character of the shear strain. This shock crossing can only be by a rarefaction shock crossing. To seek the state which is final, a branch of the curve $\sigma_H\ [v,\ v\ (M)]$ corresponding to rarefaction can be constructed on the $\sigma - v$ plane. Its point of intersection with the ray Π_σ corresponds to the final state of the second shock crossing.

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STRESS CONCENTRATION ON AN ELLIPSOIDAL INHOMOGENEITY IN AN ANISOTROPIC ELASTIC MEDIUM

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The tensor stress concentration coefficient connecting the stress on the boundary of an inhomogeneity in an anisotropic elastic medium with the external field is represented as the product of two factors. The first is universal for any inhomogeneity and depends on the elastic constants of the medium and the inhomogeneity, and on the normal to the surface. Its construction reduces to an algebraic operation of inverting a third-order matrix. The second factor is a constant tensor in the ellipsoid case, which is expressed in terms of the mean value of the first factor over the surface of the ellipsoid. Explicit formulas are obtained from the homogeneous and linear external fields. The cases of a cavity and rigid inclusion are examined separately. For an arbitrary polynomial field the problem

reduces to solving a finite system of linear algebraic equations. An investigation of the expressions obtained is simplified substantially by the fact that the ellipsoid parameters enter only in the scalar factor under the integral.

The stresses at an ellipsoidal cavity [1–4] and on an ellipsoidal inhomogeneity [5] in a homogeneous external field have been investigated under the assumption of isotropy of the external elastic medium. A spheroidal cavity in a transversally isotropic medium has been examined in [6]. Some general considerations about the case of an anisotropic medium are contained in [7]. The problem of an ellipsoid in an isotropic medium in an external linear field has been reduced in [8] to systems of algebraic equations but their solution has not been obtained. Stresses outside the ellipsoidal inhomogeneity have been examined in all these papers, and the passage to the surface was accomplished only at characteristic points (vertices of the ellipsoid).

1. The problem of stress concentration on the surface of an ellipsoidal inhomogeneity in an anisotropic medium in an external polynomial field is solved. The purpose of the paper is to represent the limit value, from outside, of the stress tensor σ_- on the boundary S of the ellipsoidal inhomogeneity as

$$\sigma_-^{\alpha\beta}(\mathbf{n}) = F^{\alpha\beta}_{\cdot\cdot\lambda\mu}(\mathbf{n})\,\sigma_0^{\lambda\mu} \tag{1.1}$$

Here σ_0 is the external polynomial field, $F(\mathbf{n})$ is an operator concentration coefficient, and $\mathbf{n}$ is the normal to S.

It is convenient to construct the operator $F(\mathbf{n})$ in two steps. First we find the relationship

$$\sigma_-^{\alpha\beta}(\mathbf{n}) = B^{\alpha\beta\rho\tau}(\mathbf{n})\,\varepsilon^+_{\rho\tau}(\mathbf{n}) \tag{1.2}$$

connecting $\sigma_-(\mathbf{n})$ with the value of the strain $\varepsilon^+(\mathbf{n})$ within the inhomogeneity at the same boundary point. The relationship is valid for an inhomogeneity of arbitrary shape, and the calculation of the tensor coefficient $B(\mathbf{n})$ which depends on the normal and the elastic characteristics of the inhomogeneity and the external medium, reduces to algebraic operations. The corresponding problem of two joined media, which might also be of independent interest, is solved in Sect. 2.

In the next step (Sect. 3), an operator B^{-1} is constructed which transforms the external field σ_0 into a field ε^+ within the inhomogeneity

$$\varepsilon^+_{\rho\tau} = B^{-1}_{\rho\tau\lambda\mu}\sigma_0^{\lambda\mu} \tag{1.3}$$

Here the property of polynomial conservativity of an ellipsoidal domain [9] is substantially used. The operator B^{-1} hence turns out to be a functional of the coefficient $B(\mathbf{n})$ (which explains the choice of notation).

Explicit expressions are found in Sect. 4 for the operator concentration coefficient

$$F^{\alpha\beta}_{\cdot\cdot\lambda\mu}(\mathbf{n}) = B^{\alpha\beta\rho\tau}(\mathbf{n})\,B^{-1}_{\rho\tau\lambda\mu} \tag{1.4}$$

for the most important cases of homogeneous and linear external fields, and its nontrivial properties are established. The calculation for an arbitrary external polynomial field reduces to quadratures and the solution of a finite system of linear algebraic equations. The limit cases of a cavity and a rigid inclusion are examined.

Specific computations of the concentration coefficient components are carried out in Sect. 5 for some particular cases of anisotropy.

2. Let us consider the problem of two joined media with the elastic moduli tensors $c_-^{\alpha\beta\lambda\mu}$ and $c_+^{\alpha\beta\lambda\mu}$. Considering the customary conditions of continuity of the displacement and the normal stress vector to be satisfied on a (sufficiently smooth) interface S

$$u_\alpha^- = u_\alpha^+, \qquad n_\alpha\sigma_-^{\alpha\beta} = n_\alpha\sigma_+^{\alpha\beta} \tag{2.1}$$

we find an expression for the jump in stress or strain at some point of the boundary S.

The continuity of the tangential component of the tensor $\nabla\mathbf{u}$ on S follows from the continuity of $\mathbf{u}$ on S. Introducing the projection operator

$$\pi_{\alpha\beta} = n_\alpha n_\beta, \qquad \vartheta_{\alpha\beta} = \delta_{\alpha\beta} - n_\alpha n_\beta \tag{2.2}$$

we write this condition as

$$\vartheta_\lambda{}^\rho \partial_\rho u_\mu^- = \vartheta_\lambda{}^\rho \partial_\rho u_\mu^+ \tag{2.3}$$

Let us transform the second of the conditions (2.1). We have $\sigma = c\varepsilon = c\nabla\mathbf{u}$, where replacement of ε by $\nabla\mathbf{u}$ is possible because of the symmetry of the elastic moduli tensor. Decomposing $\nabla\mathbf{u}^-$ into the sum of normal and tangential components, we find

$$n_\alpha c_-^{\alpha\beta\nu\gamma}\pi_\nu{}^\rho\partial_\rho u_\gamma^- + n_\alpha c_-^{\alpha\beta\nu\gamma}\vartheta_\nu{}^\rho\partial_\rho u_\gamma^- = n_\alpha c_+^{\alpha\beta\rho\tau}\partial_\rho u_\tau^+ \tag{2.4}$$

Using (2.3), we obtain an equation in $\mathbf{n}\cdot\nabla\mathbf{u}^-$

$$L_-^{\beta\gamma}(\mathbf{n})\, n^\rho\partial_\rho u_\gamma^- = L_-^{\beta\gamma}(\mathbf{n})\, n^\rho\partial_\rho u_\gamma^+ + n_\nu\,[c^{\nu\beta\rho\tau}]\,\partial_\rho u_\tau^+ \tag{2.5}$$

where

$$L_-^{\beta\gamma}(\mathbf{n}) = c_-^{\alpha\beta\nu\gamma} n_\alpha n_\nu, \qquad [c^{\nu\beta\rho\tau}] = c_+^{\nu\beta\rho\tau} - c_-^{\nu\beta\rho\tau} \tag{2.6}$$

Applying the matrix $G^-(\mathbf{n})$ the inverse of matrix $L_-(\mathbf{n})$ to both sides of (2.5) (the matrix $G^-(\mathbf{n})$ exists because of the positive-definiteness of the elastic energy), we solve (2.5) for $\mathbf{n}\cdot\nabla\mathbf{u}^-$. Then, tensor multiplication of the result by $\mathbf{n}$ yields

$$\pi_\lambda{}^\rho\partial_\rho u_\mu^- = \pi_\lambda{}^\rho\partial_\rho u_\mu^+ + n_\lambda G_{\mu\beta}^-(\mathbf{n})\, n_\nu\,[c^{\nu\beta\rho\tau}]\,\partial_\rho u_\tau^+ \tag{2.7}$$

Combining with (2.3) and symmetrizing with respect to the subscripts $\lambda\mu$, we have

$$[\varepsilon_{\lambda\mu}] = \varepsilon_{\lambda\mu}^+ - \varepsilon_{\lambda\mu}^- = -K_{\lambda\mu\nu\eta}^-[c^{\nu\eta\rho\tau}]\,\varepsilon_{\rho\tau}^+ = -K_{\lambda\mu\nu\eta}^+[c^{\nu\eta\rho\tau}]\,\varepsilon_{\rho\tau}^- \tag{2.8}$$

$$K_{\lambda\mu\nu\eta}^{\pm}(\mathbf{n}) = [n_\lambda G_{\mu\eta}^{\pm}(\mathbf{n})\, n_\nu]_{(\lambda\mu)(\nu\eta)} \tag{2.9}$$

The parentheses indicate symmetrization with respect to the appropriate subscripts.

From (2.8) we also obtain an expression for the jump in the stress $[\sigma]$ on S and the desired relationship (1.2). Henceforth, the minus and plus signs will refer, respectively, to the external medium and the inhomogeneity with the elastic moduli tensors c_0 and $c_0 + c_1$. We then have from the second equality in (2.6) and (2.8) for the coefficient $B(\mathbf{n})$ in (1.2)

$$B^{\alpha\beta\rho\tau}(\mathbf{n}) = c_0^{\alpha\beta\rho\tau} + c_0^{\alpha\beta\lambda\mu}\, \overset{\circ}{K}_{\lambda\mu\nu\eta}(\mathbf{n})\, c_1^{\nu\eta\rho\tau} \tag{2.10}$$

Here the tensor $K_0(\mathbf{n})$ is connected to the tensor $G_0(\mathbf{n}) = L_0^{-1}(\mathbf{n})$ by means of (2.9). In turn, $L_0(\mathbf{n})$ is defined by the first equality of (2.6) in which c_- should be replaced by c_0.

The connection of $G_0(\mathbf{n})$ to the Green's tensor $G_0(\mathbf{x})$ of an infinite homogeneous medium is of essential value for the sequel. Let $G_0(\mathbf{k})$ be the Fourier transform of the Green's tensor $G_0(\mathbf{x})$ (see [10], for example). It can then be said that $G_0(\mathbf{n})$ agrees with the value of $G_0(\mathbf{k})$ on the unit sphere $|\mathbf{k}| = 1$, i.e. for $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$. In the case of an isotropic medium (with Lamé coefficients $\lambda_0,\ \mu_0$) $G_0(\mathbf{n})$ and $K_0(\mathbf{n})$ have the simplest form. In particular

$$\overset{\circ}{K}_{\alpha\beta\lambda\mu}(\mathbf{n}) = \left[-\frac{\lambda_0+\mu_0}{\mu_0(\lambda_0+2\mu_0)} n_\alpha n_\beta n_\lambda n_\mu + \frac{1}{\mu_0} n_\alpha n_\lambda \delta_{\beta\mu}\right]_{(\alpha\beta)(\lambda\mu)} \tag{2.11}$$

Let us also present formulas for the strain jump $[\varepsilon]$ on the interface between two isotropic media for the case when the x^3-axis coincides with the normal $\mathbf{n}$

$$[\varepsilon_{11}] = [\varepsilon_{12}] = [\varepsilon_{22}] = 0, \quad [\varepsilon_{13}] = -\frac{[\mu]}{\mu^-}\varepsilon_{13}^+, \quad [\varepsilon_{23}] = -\frac{[\mu]}{\mu^-}\varepsilon_{23}^+$$

$$[\varepsilon_{33}] = -\frac{1}{\lambda^- + 2\mu^-}\{[\lambda](\varepsilon_{11}^+ + \varepsilon_{22}^+ + \varepsilon_{33}^+) + 2[\mu]\varepsilon_{33}^+\} \tag{2.12}$$

Let us note that in this most simple case the formulas for $[\varepsilon]$ can be obtained directly by using obvious reasoning. It follows from the first condition of (2.1), as has already been mentioned above, that the tangential component of the tensor $\nabla\mathbf{u}$ is continuous. In the coordinate system selected this means the continuity of the six components $\partial_\alpha u_\beta$ with the subscripts 1β and 2β, and consequently, the three strain components ε_{11}, ε_{12} and ε_{22}.

The continuity of the three stress components σ^{13}, σ^{23} and σ^{33} results from the second condition of (2.1). Using the conditions obtained for σ and ε as well as Hooke's law for an isotropic medium, we find expressions (2.12) for $[\varepsilon]$ and also analogous formulas for $[\sigma]$.

N o t e. Relationships for the stress and strain jumps are a result of the matching conditions (2.1), but they are not equivalent since they are satisfied under weaker assumptions. In particular, the component u_3 can have a jump on the interface for the considered case of isotropic media.

3. Let us turn to the fundamental problem of constructing the operator B^{-1} in (1.3). We consider an unbounded medium with the elastic moduli tensor $c(\mathbf{x}) = c_0 + c_1 V(\mathbf{x})$, where $\mathbf{x}(x^1, x^2, x^3)$ is a point of the medium, c_0 and c_1 are constant tensors, and $V(\mathbf{x})$ is a characteristic function of the ellipsoid (equal to unity within and zero outside the ellipsoid). The Cartesian coordinate system is connected to the ellipsoid semiaxes a^α $(\alpha = 1, 2, 3)$ given by the equation (*)

$$\mathbf{x}a^{-2}\mathbf{x} = x^\alpha (a^{-2})_{\alpha\beta} x^\beta = 1, \qquad a^{\alpha\beta} = a^\alpha \delta^{\alpha\beta} \tag{3.1}$$

The stresses or displacements at infinity and (nonzero in the general case) external forces $\mathbf{q}(\mathbf{x})$ are considered known. The equation for the displacement $\mathbf{u}(\mathbf{x})$ is

$$-\partial_\beta [c^{\alpha\beta\lambda\mu}(\mathbf{x})\, \partial_\lambda u_\mu(\mathbf{x})] = q^\alpha(\mathbf{x}) \tag{3.2}$$

or in compact notation

$$L\mathbf{u} = \mathbf{q}, \qquad L = -\nabla c \nabla \tag{3.3}$$

Understood in the sense of generalized functions, this equation will automatically assure compliance with the matching conditions (2.2) on the boundary S of the ellipsoidal inhomogeneity if it is assumed that $\mathbf{q}$ has no single or double layer type singularity on S

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*) Here and henceforth, summation is not carried out over identical super-(or sub)scripts.

It is henceforth expedient to reduce the problem to an integral equation. Let us set $\mathbf{u}(\mathbf{x}) = \mathbf{u}_0(\mathbf{x}) + \mathbf{u}_1(\mathbf{x})$, where $\mathbf{u}_0(\mathbf{x})$ is the external field, i.e. the field which would exist in a homogeneous medium ($c_1 = 0$) for given external forces $\mathbf{q}(\mathbf{x})$ and conditions at infinity, and $\mathbf{u}_1(\mathbf{x})$ is the perturbation caused by the inhomogeneity. Then $\mathbf{u}_1(\mathbf{x})$ satisfies the equation

$$(L_0 + L_1)\mathbf{u}_1 = -L_1\mathbf{u}_0, \quad L_0 = -\nabla c_0 \nabla, \quad L_1 = -\nabla c_1 \nabla \tag{3.4}$$

since $L_0\mathbf{u}_0 = \mathbf{q}$. Evidently $\mathbf{u}_1(\mathbf{x}) \to 0$ as $\mathbf{x} \to \infty$. The right side of the equation for $\mathbf{u}_1$ has a simple layer type singularity on S. It follows from potential theory that $\mathbf{u}_1$ is continuous in this case, and $\varepsilon_1 = \operatorname{def} \mathbf{u}_1$ (def is the strain operator) has a jump on S which assures the continuity of the normal stress vector.

Let $G_0 = L_0^{-1}$ be the Green's operator of a homogeneous medium whose kernel is the Green's tensor $G_{\alpha\beta}^{\circ}(\mathbf{x})$. Applying the operator $\operatorname{def} G_0$ to both sides of (3.4) and taking account of the symmetry of the tensors c_0 and c_1, we obtain an integral equation in $\varepsilon(\mathbf{x}) = \varepsilon_0(\mathbf{x}) + \varepsilon_1(\mathbf{x})$

$$\varepsilon + K_0 c_1 \varepsilon = \varepsilon_0, \quad K_0 = -\operatorname{def} G_0 \operatorname{def} \tag{3.5}$$

Here the integral operator K_0 has the kernel

$$\overset{\circ}{K}_{\alpha\beta\lambda\mu}(\mathbf{x} - \mathbf{x}') = -[\partial_\alpha \partial_\lambda \overset{\circ}{G}_{\beta\mu}(\mathbf{x} - \mathbf{x}')]_{(\alpha\beta)(\lambda\mu)} \tag{3.6}$$

Equation (3.5) is equivalent to (3.4) and because of the piecewise continuity of ε can be written as a system

$$\varepsilon^+ + K_0^+ c_1 \varepsilon^+ = \varepsilon_0^+ \tag{3.7}$$

$$\varepsilon^- = \varepsilon_0^- - K_0 c_1 \varepsilon^+ \tag{3.8}$$

Here $K_0^+ = V K_0 V$ is the contraction of the operator K_0 in the ellipsoidal domain V. The first equation defines the strain ε^+ within, and the second the continuation of the solution on the complement to V. The existence and uniqueness of the solution (3.7) follow directly from the existence and uniqueness of the solution of the equivalent equation (3.4) from which follows the existence of the operator

$$B^{-1} = (c_0 + c_0 K_0^+ c_1)^{-1} \tag{3.9}$$

The possibility of an explicit construction of the operator B^{-1} for the ellipsoidal domain V is based on the fact that it possesses the property of polynomial conservativity (the p-property); if the external field $\sigma_0(\mathbf{x})$ in the neighborhood of V is a polynomial, then the field $\varepsilon^+(\mathbf{x})$ induced within V is a polynomial of the same degree. For the particular case of a homogeneous external field this property is proved in [5, 7], and in the general case in [9]. It is shown in [9] that the calculation of ε^+ for an arbitrary external polynomial field reduces to quadratures and to the solution of a finite system of linear algebraic equations.

According to the p-property, the external homogeneous field σ_0 induces a homogeneous field ε^+ within V and consequently, the operator B^{-1} is a constant quadrivalent tensor in this case, which we denote by B_0^{-1}, i.e.

$$\sigma_0 = B_0 \varepsilon^+ \tag{3.10}$$

In order to find B_0, let us integrate (3.7) over the domain V. Taking into account that ε_0^+, ε^+ and c_1 are constant tensors, and denoting the volume of the ellipsoid by v, we have

$$\varepsilon^+ + \left[\frac{1}{v}\iint V(\mathbf{x}) V(\mathbf{x}') K_0(\mathbf{x} - \mathbf{x}')\, d\mathbf{x}\, d\mathbf{x}'\right] c_1 \varepsilon^+ = \varepsilon_0 \tag{3.11}$$

To evaluate the integral entering here, let us make the change of variable $\mathbf{x} = a\mathbf{y}$, where the matrix a is given by (3.1). Then

$$\iint V(\mathbf{x})\, V(\mathbf{x}')\, K_0(\mathbf{x}-\mathbf{x}')\, d\mathbf{x}d\mathbf{x}' = |\det a|^2 \iint V(a\mathbf{y})\, V(a\mathbf{y}')\, K_0[a(\mathbf{y}-\mathbf{y}')]\, d\mathbf{y}d\mathbf{y}'$$

Using the Parseval formula, let us reduce the double integral to a single one

$$\iint V'(\mathbf{y})\, V'(\mathbf{y}')\, K_0'(\mathbf{y}-\mathbf{y}')\, d\mathbf{y}d\mathbf{y}' = \frac{1}{8\pi^3}\int V'^2(\mathbf{k})\, K_0'(\mathbf{k})\, d\mathbf{k} \qquad (3.12)$$

where $V'(\mathbf{k})$ and $K_0'(\mathbf{k})$ are the Fourier transforms $V'(\mathbf{y}) = V(a\mathbf{y})$ and $K_0'(\mathbf{y}) = K_0(a\mathbf{y})$. Since $V'(\mathbf{y})$ is the characteristic function ot a unit sphere, its Fourier transform $V'(\mathbf{k})$ depends only on $|\mathbf{k}|$. By direct calculation we find

$$V'(\mathbf{k}) = \frac{4\pi}{|\mathbf{k}|^3}(\sin|\mathbf{k}| - |\mathbf{k}|\cos|\mathbf{k}|) \qquad (3.13)$$

The function $K_0'(\mathbf{y})$ is homogeneous of zero degree, and therefore, $K_0'(\mathbf{k})$ depends only on the unit vector $\boldsymbol{\omega} = \mathbf{k}/|\mathbf{k}|$

$$K_0'(\mathbf{k}) = |\det a|^{-1} K_0(a^{-1}\mathbf{k}) = |\det a|^{-1} K_0(a^{-1}\boldsymbol{\omega}) \qquad (3.14)$$

This permits separate integration over $|\mathbf{k}|$ and over $\boldsymbol{\omega}$ in (3.12). By virtue of the homogeneity of $K_0(\mathbf{k})$, we have for the integral over the unit sphere

$$\int K_0(a^{-1}\boldsymbol{\omega})\, d\boldsymbol{\omega} = \int K_0\left(\frac{a^{-1}\boldsymbol{\omega}}{|a^{-1}\boldsymbol{\omega}|}\right) d\boldsymbol{\omega} = |\det a| \int K_0(\mathbf{n})\, \rho^3(\mathbf{n})\, d\mathbf{n} \qquad (3.15)$$

$$\mathbf{n} = \frac{a^{-1}\boldsymbol{\omega}}{|a^{-1}\boldsymbol{\omega}|}, \qquad \rho(\mathbf{n}) = \frac{1}{|a\mathbf{n}|} = \frac{1}{\sqrt{\mathbf{n}a^2\mathbf{n}}} \qquad (3.16)$$

Let us introduce some notation. Let f be a function given on the ellipsoid S. A point $\mathbf{x}$ of the ellipsoid and the normal $\mathbf{n}$ to it are connected by the relationships

$$\mathbf{x} = \frac{a^2\mathbf{n}}{\sqrt{\mathbf{n}a^2\mathbf{n}}}, \qquad \mathbf{n} = \frac{a^{-2}\mathbf{x}}{\sqrt{\mathbf{x}a^{-4}\mathbf{x}}} \qquad (3.17)$$

and, therefore, f can be considered as a function of the normal $\mathbf{n}$. We define the mean value of $f(\mathbf{n})$ by the expression

$$\langle f(\mathbf{n})\rangle = \frac{|\det a|}{4\pi}\int f(\mathbf{n})\, \rho^3(\mathbf{n})\, d\mathbf{n} \qquad (3.18)$$

where $\rho(\mathbf{n})$ is given by (3.16).

The desired tensor B_0 connecting σ_0 and ε^+ in (3.10) can then be represented as the mean value of the tensor $B(\mathbf{n})$ over the ellipsoid, determined by the matching problem. Indeed, by multiplying (3.11) on the left by c_0 and integrating taking account of (3.12), (3.13), (3.15) and (3.18), we finally find

$$B_0^{\alpha\beta\lambda\mu} = \langle B^{\alpha\beta\lambda\mu}(\mathbf{n})\rangle \qquad (3.19)$$

Here $c_1 = -c_0$ and $c_1 \to \infty$ correspond to the limit cases of a cavity and rigid inclusion. It can be shown that B_0^{-1} exists for both these cases.

Now, let the external field be linear, i.e. $\sigma_0^{\lambda\mu} = d^{\lambda\mu\tau}x_\tau$. On the basis of the p-property $\varepsilon_{\alpha\beta}^+(\mathbf{x})$ is a linear homogeneous function $\varepsilon_{\alpha\beta}^+(\mathbf{x}) = b_{\alpha\beta\nu}x^\nu$. In this case the operator B^{-1} reduces to a constant hexavalent tensor connecting the constant tensors b and d (symmetric on the first pair of indices). It is convenient to represent this relation as

$$d^{\lambda\mu\tau} = 3B_1^{\lambda\mu\tau\alpha\beta\eta}(a^2)_\eta{}^\nu b_{\alpha\beta\nu} \tag{3.20}$$

To find B_1 we perform tensor multiplication of (3.7) by $\mathbf{x}$ and integration over V. Calculations analogous to those carried out above for the homogeneous field yield

$$B_1^{\alpha\beta\eta\lambda\mu\tau} = \langle \rho(\mathbf{n})\, n^\eta B^{\alpha\beta\lambda\mu}(\mathbf{n})\, n^\tau \rho(\mathbf{n})\rangle \tag{3.21}$$

Evidently the tensor B_1 is symmetric within each pair of superscripts $\alpha\beta$, $\lambda\mu$, $\eta\tau$. Comparison of (3.19) with (3.21) shows that B_0 and B_1 can be interpreted as corresponding moments of B ($\mathbf{n}$).

Let us note that in contrast to the case of a homogeneous external field, the passage to a cavity requires some care in the inversion of the tensor B_1. This is related to the formal existence of external linear fields σ_0 (x) (which do not satisfy the equilibrium condition $\operatorname{div}\sigma = 0$), which are not perturbed by the inhomogeneity. In the case of a cavity the tensor B_1^{-1} exists only in a subspace not including such fields. No detailed investigation of this question is made here. If $\sigma_0(\mathbf{x})$ is polynomial of higher degree, then the properties of the operator K_6^+ examined in [9] permit the finding of linear recurrent algebraic relations from the integral equation (3.7), which would relate the coefficients of the polynomials σ_0 (x) and ε^+ (x), and the construction of an explicit expression for the operator B, analogously to the cases presented above.

4. The operator concentration coefficient F ($\mathbf{n}$) is defined by the relationship (1.1) and in conformity with (1.4) is expressed in terms of the tensor B ($\mathbf{n}$) and the operator B^{-1}.

Let the external field be homogeneous. Then, as has been shown above, the operator B^{-1} in (1.4) agrees with the constant tensor B_0^{-1} and the concentrations coefficient F_0 ($\mathbf{n}$) to the quadrivalent tensor

$$F_{0\cdot\cdot\lambda\mu}^{\alpha\beta}(\mathbf{n}) = B^{\alpha\beta\sigma\tau}(\mathbf{n})\,\langle B(\mathbf{n})\rangle^{-1}_{\sigma\tau\lambda\mu} \tag{4.1}$$

This representation is especially convenient for the investigation of the dependence of the concentration coefficient on the shape of the ellipsoid. Indeed, the local dependence of F_0 ($\mathbf{n}$) on the normal to the ellipsoid (or equivalently, on the point) is given by the first factor B ($\mathbf{n}$), which is independent of the ellipsoid parameters and always remains finite. The second factor B_0^{-1}, which has a singularity in the case of the passage to the limit to the crack, yields a fundamental contribution to the stress concentration.

It follows from (3.18) and (3.19) that the whole dependence of B_0 on the ellipsoid parameters is concentrated in the scalar weight factor ρ ($\mathbf{n}$). This circumstance is quite essential since it permits a complete investigation of the stress concentration in the limit cases of an ellipsoidal crack and a needle for an arbitrary anisotropic medium by a series expansion of ρ ($\mathbf{n}$) in a small parameter.

A curious relationship holds for the mean values $\langle F_0(\mathbf{n})\rangle$ and $\langle\sigma_-(\mathbf{n})\rangle$. We find from (3.19) and (4.1)

$$\langle F_0(\mathbf{n})\rangle = I_0 \tag{4.2}$$

where I_0 is a unit quadrivalent tensor. It then follows from (1.1) that

$$\langle \sigma_-(\mathbf{n})\rangle = \sigma_0 \tag{4.3}$$

It is necessary to set $c_1 \to \infty$ for a rigid inclusion. In the limit we have

$$F_0(\mathbf{n}) = c_0 K_0(\mathbf{n})\,\langle c_0 K_0(\mathbf{n})\rangle^{-1}$$

Let us examine the linear external field $\sigma_0^{\lambda\mu}(\mathbf{x}) = d^{\lambda\mu\tau}x_\tau$. It can be shown that $d^{\lambda\mu\tau} = 3\langle\sigma_0^{\lambda\mu}(\mathbf{n})\, n^\tau\rho(\mathbf{n})\rangle$. Then taking account of (1.2), (3.20) and (3.21), as well as the connection (3.14) between $\mathbf{x}$ and $\mathbf{n}$ on the boundary of the ellipsoid, we find

$$\sigma_-^{\alpha\beta}(\mathbf{n}) = \rho(\mathbf{n})\, n^\nu F_{1\cdot\cdot\nu\lambda\mu\tau}^{\alpha\beta}(\mathbf{n})\langle\sigma_0^{\lambda\mu}(\mathbf{n})\, n^\tau\rho(\mathbf{n})\rangle \tag{4.4}$$

$$F_{1\cdot\cdot\nu\lambda\mu\tau}^{\alpha\beta}(\mathbf{n}) = B^{\alpha\beta\varkappa\rho}(\mathbf{n})\langle\rho(\mathbf{n})\,\mathbf{n}B(\mathbf{n})\,\mathbf{n}\rho(\mathbf{n})\rangle_{\varkappa\rho\nu\lambda\mu\tau}^{-1}$$

In particular, for a rigid inclusion

$$F_1(\mathbf{n}) = c_0K_0(\mathbf{n})\langle\rho(\mathbf{n})\,\mathbf{n}c_0K_0(\mathbf{n})\,\mathbf{n}\rho(\mathbf{n})\rangle^{-1}$$

For the mean values $\langle\rho(\mathbf{n})\,\mathbf{n}F_1(\mathbf{n})\,\mathbf{n}\rho(\mathbf{n})\rangle$ and $\langle\sigma_-(\mathbf{n})\,\mathbf{n}\rho(\mathbf{n})\rangle$ there are valid formulas analogous to (4.2) and (4.3). In conformity with (4.4) we have

$$\langle\rho(\mathbf{n})\,\mathbf{n}F_1(\mathbf{n})\,\mathbf{n}\rho(\mathbf{n})\rangle = I_1$$

$$\langle\sigma_-(\mathbf{n})\,\mathbf{n}\rho(\mathbf{n})\rangle = \langle\sigma_0(\mathbf{n})\,\mathbf{n}\rho(\mathbf{n})\rangle$$

where I_1 is a unit hexavalent tensor.

5. As an illustration, let us calculate the concentration coefficient $F_0(\mathbf{n})$ for an ellipsoidal cavity in an external homogeneous field. Let us examine the simplest case of anisotropy which is obtained by tension along the three axes of an ellipsoid, i.e.

$$c_0^{\alpha\beta\lambda\mu} = \mu_0(g^{\alpha\lambda}g^{\beta\mu} + g^{\alpha\mu}g^{\beta\lambda}) + \frac{2\mu_0\nu_0}{1-2\nu_0}g^{\alpha\beta}g^{\lambda\mu}$$

$$g = h^2, \qquad h^{\alpha\beta} = h^\alpha\delta^{\alpha\beta} \tag{5.1}$$

Here ν_0 is the Poisson's ratio.

Let us perform the coordinate transformation setting $\mathbf{x}' = h^{-1}\mathbf{x}$. Then the tensor of the elastic constants c_0' will agree with the isotropic tensor, and the equation of the ellipsoid becomes

$$\mathbf{x}'b^{-2}\mathbf{x}' = 1, \qquad b^{-2} = ha^{-2}h$$

Thus, it is sufficient to calculate the concentration coefficient $F_0'(\mathbf{n}')$ for an isotropic medium. The components of $F_0(\mathbf{n})$ are related to the components of $F_0'(\mathbf{n}')$ by means of

$$F_{0\cdot\cdot\lambda\mu}^{\alpha\beta}(\mathbf{n}) = \frac{h^\alpha h^\beta}{h^\lambda h^\mu}F_{0\cdot\cdot\lambda\mu}'^{\alpha\beta}(h\mathbf{n}\rho(\mathbf{n}))$$

Let us turn to the calculation of $F_0')(\mathbf{n}')$ and let us henceforth omit the primes. According to (4.1), $F_0(\mathbf{n})$ consists of two factors. The first, the tensor $B(\mathbf{n})$, depends on the normal to the ellipsoid and is defined by (2.10) in which should be set $c_1 = -c_0$. for a cavity. Substituting (2.11) into (2.10) and using the expression for an isotropic tensor of the elastic moduli (which agrees with (5.1)), we find when $g^{\alpha\beta}$ is replaced by $\delta^{\alpha\beta}$ $(\varkappa_0 = 2\mu_0 / 1 - \nu_0)$

$$B^{\alpha\beta\sigma\tau}(\mathbf{n}) = \varkappa_0[\nu_0\delta^{\alpha\beta}\delta^{\sigma\tau} - n^\alpha n^\beta\delta^{\sigma\tau} - n^\sigma n^\tau\delta^{\alpha\beta}) + \frac{1-\nu_0}{2}(\delta^{\alpha\sigma}\delta^{\beta\tau} + \delta^{\alpha\tau}\delta^{\beta\sigma} -$$
$$n^\alpha n^\sigma\delta^{\beta\tau} - n^\alpha n^\tau\delta^{\beta\sigma} - n^\beta n^\sigma\delta^{\alpha\tau} - n^\beta n^\tau\delta^{\alpha\sigma}) + n^\alpha n^\beta n^\sigma n^\tau]$$

It is seen that $B(\mathbf{n})$ has the symmetry c_0, i.e. is symmetric within the pairs and in commutation of the pairs of indices (in the general case of an inhomogeneous inclusion there is no symmetry in the commutation of pairs). As a result the second member in $F_0(\mathbf{n})$, the constant tensor B_0^{-1}, has the symmetry c_0. To calculate it the tensor B_0

given by (3.18) and having the structure of an orthorhombic tensor with nine nonzero essential components should be inverted. Direct calculations yield

$$B_0^{1111} = \varkappa_0 \left[1 - \frac{1}{8\pi}(3I_{11} + I_1)\right] \tag{5.2}$$

$$B_0^{1122} = \varkappa_0 \left\{\nu_0 - \frac{1}{16\pi}[I_{12} + I_{21} - (1 - 4\nu_0)(I_1 + I_2)]\right\}$$

$$B_0^{1212} = \varkappa_0 \left\{\frac{1-\nu_0}{2} - \frac{1}{16\pi}[I_{12} + I_{21} + (1 - 2\nu_0)(I_1 + I_2)]\right\}$$

$$I_p = \frac{3}{2} v \int_0^\infty \frac{d\xi}{(b_p^2 + \xi)\Delta(\xi)}, \qquad I_{pq} = \frac{3}{2} v b_p^2 \int_0^\infty \frac{d\xi}{(b_p^2 + \xi)(b_q^2 + \xi)\Delta(\xi)}$$

$$\Delta(\xi) = \sqrt{(b_1^2 + \xi)(b_2^2 + \xi)(b_3^2 + \xi)}$$

The integrals I_p, I_{pq} are expressed in terms of elliptic integrals of the first and second kind. The remaining six linearly independent components of the tensor B_0 are obtained from (5.2) by a cyclic replacement of the superscripts 1, 2, 3. In the particular case of an ellipsoid of revolution $(b_1 = b_2)$ only six essential components remain in the tensor B_0 and the elliptic integrals vanish

$$B_0^{1111} = \frac{\varkappa_0}{8}(3 + 2f_1 + 3f_2), \quad B_0^{3333} = \varkappa_0(1 - 2f_1 + f_2)$$

$$B_0^{1122} = \frac{\varkappa_0}{8}[1 - 2(1 - 4\nu_0)f_1 + f_2], \quad B_0^{1133} = \frac{\varkappa_0}{2}[\nu_0 + (1 - \nu_0)f_1 - f_2] \tag{5.3}$$

$$B_0^{1212} = \frac{\varkappa_0}{8}[1 + 2(1 - 2\nu_0)f_1 + f_2], \quad B_0^{1313} = \frac{\varkappa_0}{4}[1 - \nu_0 + (1 + \nu_0)f_1 - 2f_2]$$

$$f_1 = \frac{1}{1 - \alpha^2} - \frac{\alpha}{(1 - \alpha^2)^{3/2}} \arcsin\sqrt{1 - \alpha^2}$$

$$f_2 = \frac{2 + \alpha^2}{2(1 - \alpha^2)^2} - \frac{3\alpha}{(1 - a^2)^{5/2}} \arcsin\sqrt{1 - \alpha^2}, \quad \alpha = \frac{b_3}{b_1}$$

As is customary, in order to evaluate B_0^{-1} we represent the quadrivalent tensor B_0 as a sixth order matrix decomposed into two blocks of third order matrices. The components $B_0^{\lambda\lambda\mu\mu}$ form a symmetric matrix, and the doubled members $B_0^{\lambda\mu\lambda\mu}$ $(\lambda \neq \mu)$ – a diagonal matrix, and consequently

$$(B_0^{-1})_{\lambda\mu\lambda\mu} = \frac{1}{4B_0^{\lambda\mu\lambda\mu}} \qquad (\lambda \neq \mu) \tag{5.4}$$

Let us examine two cases.

1. The external field $\sigma_0^{\alpha\beta}$ is tension along the coordinate axes, i.e. only the diagonal components $\sigma_0^{\lambda\lambda}$ are nonzero. Then the stress concentration is defined by the components

$$F_{0\cdot\cdot\lambda\lambda}^{\alpha\beta}(\mathbf{n}) = B^{\alpha\beta 11}(\mathbf{n})(B_0^{-1})_{11\lambda\lambda} + B^{\alpha\beta 22}(\mathbf{n})(B_0^{-1})_{22\lambda\lambda} + B^{\alpha\beta 33}(\mathbf{n})(B_0^{-1})_{33\lambda\lambda}$$

2. The external field $\sigma_0^{\alpha\beta}$ is pure shear. The stress concentration is determined by the components $F_{0\ldots\lambda\mu}^{\alpha\beta}(\mathbf{n})$ for $\lambda \neq \mu$, for which we have according to (4.1) and (5.3)

$$F_{0\cdot\cdot\lambda\mu}^{\alpha\beta}(\mathbf{n}) = \frac{B^{\alpha\beta\lambda\mu}(\mathbf{n})}{2B_0^{\lambda\mu\lambda\mu}} \qquad (\lambda \neq \mu) \tag{5.5}$$

Thus, an investigation of the stress concentration in this case is considerably simpler than in the preceding case.

A comparison with the stress concentration coefficients at the vertex of an ellipsoid, obtained in [2], shows that the corresponding components from (5.5) yield the same values at these points. It is however essential that knowledge of the concentration coefficient just at the vertices of the ellipsoid is not sufficient since it can result in incorrect qualitative deductions. For example, for pure shear with the components σ_0^{13} at the vertex (1, 0, 0) the stress tensor $\sigma_-^{\alpha 3}$ equals zero, but there is a stress concentration in direct proximity to this vertex. Hence, a complete investigation of the stresses on the whole ellipsoid surface is necessary, and not only at its characteristic points.

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